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Asymptotic Approach to Hartmann-Poiseuille Flows[1]

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ABSTRACT

The steady state flow is considered of a viscous conducting liquid in the inlet region of a straight channel with or without a transverse magnetic field, particularly the rate of approach of the flow to the Hartmann–Poiseuille patterns downstream. As usual R, R_m, M denote the Reynolds number, magnetic Reynolds number, and Hartmann number respectively. When $M = 0$ the approach to Poiseuille flow is monotone for $R > 16.92\ldots$, but spatial oscillations occur for smaller R. In general the approach to the limiting pattern is exponential and the dependence of the exponents on R, R_m, M is analysed. The exponents fall into two classes, one depending chiefly on R, M and the other on R_m, M. For $R \gg 1$ the leading exponent is found to be approximately $-(37.6.. + 2M^2)\, R^{-1}$.

1. INTRODUCTION

We consider the steady state two-dimensional flow of a homogeneous, incompressible, viscous, conducting fluid in the inlet region of a straight channel in the presence of a transverse magnetic field. The flow will be assumed symmetric about the centre line of the channel. It is assumed that, at a great distance from the inlet, the flow tends to the Hartmann pattern. The purpose of this paper is to study the rate of approach to this limiting form as a function of the various physical parameters. In the absence of a magnetic field the problem becomes very much simpler and the Hartmann pattern reduces to the familiar parabolic flow. The introduction of an infinitesimally weak field introduces new terms into the solution which may lead to an actual discontinuity in the rate of approach to the limit.

The chief motivation for this study was to check the accuracy of a numerical solution obtained in [2] for the flow over the entire inlet region.

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In paragraph 6 we shall, where possible, compare the results obtained by the methods of this paper with those deducible from the complete numerical solution. The agreement is seen to be very good.

Sparrow, Lin and Lundgren [5] have recently investigated the problem of inlet flow in the absence of a magnetic field. They were concerned with large Reynolds numbers and limited their discussion to the boundary layer equations. Subject to some additional simplifying assumptions they incidentally obtained quantitative results concerning the asymptotic behaviour of the flow. Snyder extended their methods to the magnetohydrodynamic problem.

We should also mention earlier work by Shercliff [3] and Roidt and Cess [6], for the limiting case of large Reynolds numbers. The first of these authors simplified the treatment by assuming the problem equivalent to that of a linear transient. The latter authors attacked the boundary layer equations directly. No details are given as to their method of computation.

A comparison of our results with those of these workers will be made below in paragraph 5.2.

2. The Differential Equations

Suppose that the width of the channel is $2a$, the average longitudinal velocity u_0, the kinematic viscosity ν, the fluid density ρ, electrical conductivity σ and magnetic permeability μ. We define

$$R = \frac{2au_0}{\nu} \text{ [Reynolds' number]}$$

$$R_m = 2a\mu\sigma u_0 \text{ [Magnetic Reynolds' number]}$$

$$B_0 = \frac{1}{a}\sqrt{\frac{\rho\nu}{\sigma}}.$$

We now go over to dimensionless coordinates (x, y, z), measuring lengths in units of a, and fluid velocities (u, v, w) in units of u_0. The walls of the channel can now be taken as $y = \pm 1$.

The governing equations can be written (see [2])

$$\psi_y\beta_x - \psi_x\beta_y = E_3 + \frac{2}{R_m}\nabla^2\beta \tag{2.1}$$

$$\psi_y\nabla^2\psi_x - \psi_x\nabla^2\psi_y = \frac{2}{R}\nabla^4\psi + \frac{4}{RR_m}(\beta_y\nabla^2\beta_x - \beta_x\nabla^2\beta_y). \tag{2.2}$$

Here ψ is the hydrodynamic stream function ($u = \psi_y$, $v = -\psi_x$) and β the z-

component of the magnetic vector potential in units of aB_0. E_3 is the z-component of the electrostatic field, in units of u_0B_0, and is known to be a constant in two-dimensional flows.

We suppose that the external magnetic field is constant and in the y-direction and that the magnetic permeability of the walls is finite. It follows that, for $y = \pm 1$, β must be proportional to x. We can therefore write the following boundary conditions:

$$\psi(x, 1) = 1; \qquad \psi(x, -1) = -1 \tag{2.3}$$

$$\psi_y(x, \pm 1) = 0 \tag{2.4}$$

$$\beta(x, \pm 1) = -Mx \tag{2.5}$$

$$E_3 = -M \tag{2.6}$$

where M is a constant (the Hartmann number). We might mention that (2.6) is actually equivalent to assuming that the total net current perpendicular to the plane of flow is zero. (See Section II(d) in [2]).

The solution of (2.1), (2.2) requires additional boundary conditions, namely those at the inlet ($x = 0$, say. See [2]). However, for the purposes of this paper, (2.3)–(2.6) are all that will be needed, except for one additional point. It is clear from (2.1), (2.2) and the boundary conditions (2.3)–(2.6) that should $\beta(0, y)$ be an even function of y and $\psi(0, y)$, $\psi_x(0, y)$ both odd functions of y, then these properties will be conserved for all $x > 0$. We shall in fact make this additional assumption about the set-up at the inlet.

Our boundary conditions represent an arbitrary choice out of an infinite manifold of possible conditions. They were chosen since they correspond to those used in our original numerical solution of the complete equations [2]. Any other set of boundary conditions would have led to a similar treatment except that the function obtained for $\Delta(\alpha)$ [See para. 4 below] would have been different.

3. Behaviour at Infinity

We shall make the physical assumption that the flow field and magnetic field both tend to limiting forms as x tends to infinity. These will necessarily be of Hartmann type. It follows that

$$\psi(\infty, y) = K\left[y \cosh M - \frac{1}{M} \sinh My\right] \tag{3.1}$$

and that, for large, x,

$$\begin{aligned} \beta(x, y) &\sim \frac{KR_m}{2}\left[\frac{1}{M} \cosh My - \frac{1}{2} y^2 \sinh M\right] - Mx \\ &= \beta_\infty(x, y) \qquad \text{(say)} \end{aligned} \tag{3.2}$$

where

$$K = \left(\cosh M - \frac{1}{M}\sinh M\right)^{-1}.$$

Since it is our purpose to study the flow and magnetic field for large x we shall write

$$\psi(x, y) = \psi(\infty, y) + G(x, y) \tag{3.4}$$

$$\beta(x, y) = \beta_\infty(x, y) + H(x, y) \tag{3.5}$$

where we assume that squares and higher powers of G, H and their derivatives may be neglected. We now substitute from (3.4), (3.5) into (2.1), (2.2) and obtain the equations

$$\begin{aligned} &-MG_y + K(\cosh M - \cosh M_y)\, H_x \\ &\quad - \frac{R_m K}{2}(\sinh My - y \sinh M)\, G_x = \frac{2}{R_m}(H_{xx} + H_{yy}) \end{aligned} \tag{3.6}$$

$$\begin{aligned} &\frac{2}{R}\nabla^4 G - K(\cosh M - \cosh My)\, \nabla^2 G_x - M^2 K \cosh My \cdot G_x \\ &\quad + \frac{2K}{R}(\sinh My - y \sinh M)\, \nabla^2 H_x + \frac{4M}{RR_m}\nabla^2 H_y - \frac{2M^2K}{R}\sinh My \cdot H_x = 0. \end{aligned} \tag{3.7}$$

We shall seek a solution of (3.6), (3.7) in the form

$$G(x, y) = e^{\alpha x} g(y) \tag{3.8}$$

$$H(x, y) = e^{\alpha x} h(y). \tag{3.9}$$

Making the necessary substitutions we get

$$\begin{aligned} &-Mg'(y) + \alpha K(\cosh M - \cosh My)\, h(y) \\ &\quad + \frac{\alpha K R_m}{2}(y \sinh M - \sinh My)\, g(y) = \frac{2}{R_m}[h''(y) + \alpha^2 h(y)] \end{aligned} \tag{3.10}$$

$$\begin{aligned} &\frac{2}{R}[g''''(y) + 2\alpha^2 g''(y) + \alpha^4 g(y)] - \alpha K(\cosh M - \cosh My)[g''(y) + \alpha^2 g(y)] \\ &\quad - \alpha K M^2 \cosh My \cdot g(y) + \frac{2\alpha K}{R}(\sinh My - y \sinh M)[h''(y) + \alpha^2 h(y)] \\ &\quad + \frac{4M}{RR_m}[h''(y) + \alpha^2 h'(y)] - \frac{2\alpha M^2 K}{R}\sinh My \cdot h(y) = 0. \end{aligned} \tag{3.11}$$

According to our boundary conditions, we require solutions of (3.10), (3.11) such that g and h are symmetric and antisymmetric functions, respectively, and such that

$$g(1) = g'(1) = h(1) = 0. \tag{3.12}$$

The values of α for which a solution is possible (other than $g \equiv h \equiv 0$) constitute a discrete set. We are interested only in values of α with negative real parts and it is clear that the ultimate rate at which the flow tends to its limiting value is determined by the root or roots, the absolute value of whose negative real part is least, i.e. by the dominant eigenvalue. We shall denote by $\{\alpha_n\}$ the sequence of eigenvalues with negative real part arranged in non-increasing order of real part.

4. Determination of Eigenvalues

In accordance with our requirements above, we seek a solution of the form

$$g(y) = \sum_{n=0}^{\infty} \gamma_n y^{2n+1} \tag{4.1}$$

$$h(y) = \sum_{n=0}^{\infty} \eta_n y^{2n} \tag{4.2}$$

Substituting (4.1), (4.2) into (3.10), we obtain a recurrence relation

$$\eta_{n+1} = P_{n+1}(\gamma_0, \ldots, \gamma_n, \eta_0, \ldots, \eta_n, \alpha). \tag{4.3}$$

Similarly substitution into (3.11) yields the relation

$$\gamma_{n+1} = Q_{n+1}(\gamma_0, \ldots, \gamma_n, \gamma_0, \ldots, \eta_n, \eta_{n+1}, \alpha). \tag{4.4}$$

The functions P_{n+1}, Q_{n+1} are known and can be written down explicitly. It is easily seen in this way that the functions $g(y)$, $h(y)$ are uniquely determined if η_0, γ_0, γ_1, α are known. We can accordingly write

$$g(y) = g(\eta_0, \gamma_0, \gamma_1, \alpha, y) \tag{4.5}$$

and

$$h(y) = h(\eta_0, \gamma_0, \gamma_1, \alpha, y). \tag{4.6}$$

Since (3.10), (3.11) are linear and homogeneous we can express the general solution for given α as a linear combination of three particular solutions. We take

$$\begin{aligned}
g_1(\alpha, y) &= g(1, 0, 0, \alpha, y), & h_1(\alpha, y) &= h(1, 0, 0, \alpha, y)\\
g_2(\alpha, y) &= g(0, 1, 0, \alpha, y), & h_2(\alpha, y) &= h(0, 1, 0, \alpha, y)\\
g_3(\alpha, y) &= g(0, 0, 1, \alpha, y), & h_3(\alpha, y) &= h(0, 0, 1, \alpha, y).
\end{aligned}$$

The general solution can then be written

$$g(y) = \sum_{i=1}^{3} A_i g_i(\alpha, y) \tag{4.7}$$

$$h(y) = \sum_{i=1}^{3} A_i h_i(\alpha, y). \tag{4.8}$$

Substituting (4.7), (4.8) into (3.12) gives us three homogeneous linear equations for A_1, A_2, A_3 with determinant, $\Delta(\alpha)$, depending only on α. We have therefore to solve the equation

$$\Delta(\alpha) = 0. \tag{4.9}$$

To compute $\Delta(\alpha)$ for a given α we used recurrence relations (4.3)–(4.4), programmed for an electronic computer, which also executed the necessary summations.

The general method for solving (4.9) was first to find the approximate location of a root in the complex α-plane by plotting the level lines of Re $\Delta(\alpha)$ and Im $\Delta(\alpha)$. Following this first approximation the root was located more exactly by successive application of the method of regula falsi. Once roots had been obtained corresponding to a set of values of the physical parameters they were taken as first approximations to the roots corresponding to neighbouring values of the parameters. This helped to speed the systematic search for the eigenvalues. Indeed it rendered the computing time for each new set of roots effectively negligible. All of this work was executed on the 1604-*A* computer of Control Data Corporation at the Weizmann Institute of Science. The results are discussed in paragraph 5 below.

5. Numerical Results

It might help to clarify the description of our results if we begin with some particular cases in which (4.9) becomes simpler and can be treated by special methods.

5.1 $M = 0$ (Hydrodynamic Case)

In this case the eigenvalues depend only on R, and some of their values as a function of R are shown in Table I. Its determination was comparatively simple since only $\psi(x, y)$ had to be considered and only one differential equation (5.1) was needed with boundary conditions $g(1) = g'(1) = 0$.

$$g'''' + 2\alpha^2 g'' + \alpha^4 g - \tfrac{3}{4}\alpha R\{(1 - y^2)(g'' + \alpha^2 g) + 2g\} = 0. \tag{5.1}$$

TABLE I

THE TWO LEADING EXPONENTS AS FUNCTION OF R FOR THE SPECIAL CASE $M = 0$ (HYDRODYNAMIC CASE)

R	α_1	$\alpha_1 . R$	α_2	$\alpha_2 . R$
0	$-3.748838 \pm 1.384339i$		$-6.94998 \pm 1.67610i$	
0.5	$-3.67669 \pm 1.39141i$		$-6.88416 \pm 1.67612i$	
5	$-3.15444 \pm 1.29660i$		$-6.36088 \pm 1.52047i$	
10	$-2.80083 \pm 1.0073i$		$-5.9359 \pm 1.0082i$	
$R_c = 16.92126$	-2.63200	-44.5029		
20	-1.90100	-38.20200	$-3.37034 \pm 0.5169i$	
50	-0.749171	-37.4585	$-2.667 \pm 0.257i$	
100	-0.375787	-37.5787	-1.15391	-115.391
200	-0.188075	-37.6150	-0.57535	-115.070
500	-0.0752515	-37.6257	-0.23010	-115.05
1000	-0.0376270	-37.6270	-0.115045	-115.045
$R \to \infty$		-37.62779		-115.0453

In Table I, we note that $\alpha_1 R$, $\alpha_2 R$ change hardly at all for $R \geqslant 100$. Indeed boundary layer theory would require αR to be sensibly constant for large R. The entry corresponding to "$R \to \infty$" was, in fact, obtained by setting $a = \alpha R$ in (5.1) and letting R tend to infinity.

The value $R_c[=16.92126]$ was found to be critical in the sense that α_1 was real for $R \geqslant R_c$ and complex for $R < R_c$. The implied existence of spatial oscillations for flow at high viscosity would at first sight seem contrary to what might have been expected. However one should bear in mind that here it is precisely the viscosity which constitutes the "restoring force" while the "resistance" to it stems from the inertial forces.

5.2 LARGE VALUES OF R

The magnetohydrodynamic case of very large R can also be discussed separately. We multiply (3.11) by R and then set $\alpha = \mathrm{a}/R$ in both this equation and (3.10). Ignoring terms of order $1/R$ in both equations we obtain

$$\frac{2}{R_m} h'' + Mg' = 0 \tag{5.2}$$

$$\frac{2M}{R_m} h''' + g'''' - \frac{\mathrm{a}K}{2} [\cosh M - \cosh My]\, g'' - \frac{\mathrm{a}KM^2 \cosh mY}{2} g = 0. \tag{5.3}$$

Eliminating h we get

$$g'''' - M^2 g'' - \frac{\mathrm{a}K}{2} [\cosh M - \cosh My]\, g'' - \frac{\mathrm{a}KM^2 \cosh My}{2} g = 0 \quad (5.4)$$

Again we seek a so that there shall be a nontrivial odd solution of this equation with $g(1) = g'(1) = 0$. The entire process is similar to that already described and we note that the only relevant physical parameter is M. We show in Table II $\mathrm{a}_i(i = 1, 2)$ as a function of M, $\mathrm{a}_i(M)$ say. It appears from this table that $\mathrm{a}_1(M)$ can be approximated very closely by a linear function of M^2 and the fourth column of the table shows that, for large R, we may write

$$\alpha_1 \approx -\frac{37.62779 + 2M^2}{R}. \quad (5.5)$$

TABLE II

LIMITING BEHAVIOUR OF LEADING EXPONENTS FOR LARGE R

M	$\mathrm{a}_1(M) = \lim_{R\to\infty} \alpha_1 R$	$\mathrm{a}_2(M) = \lim_{R\to\infty} \alpha_2 R$	$-\frac{\mathrm{a}_1(M) - \mathrm{a}_1(0)}{M^2}$
0	−37.62779	−115.0453	—
0.05	−37.63297	−115.0527	2.07
0.1	−37.6484	−115.0720	2.06
1	−39.6847	−117.6713	2.06
2	−45.7976	−125.1978	2.04
3	−55.8165	−136.9069	2.02
5	−86.9517	−170.9687	1.97
10	−228.9460	−316.9010	1.91
15	−467.773	−556.598	1.91
20	−806.463	−894.758	1.92
25	−1245.45	−1332.83	1.93
30	−1784.69	−1871.16	1.96

We have not been able to find a complete analytical justification for the formula (5.5). However the following observations seem relevant.

When M is small we may develop (5.4) in powers of M [cf. (5.10), below] and see at once that a is of the form $A + BM^2 + O(M^4)$. Again for $M \gg 1$ the limiting form of (5.4) becomes

$$g'''' - \left(M^2 + \frac{\mathrm{a}}{2}\right) g'' = 0$$

leading to the series of eigenvalues $-8n^2\pi^2 - 2M^2$ $(n = 0, 1, 2, \ldots)$.

In the work of Sparrow, Lin and Lundgren [5] on the simplified Navier–Stokes equations they obtained the set of eigenvalues

$$\mathrm{a}_n = -2\theta_n{}^2\, 2\theta_n{}^2 \qquad (n = 1, 2, ...) \tag{5.6}$$

where the θ_n's are the successive roots of

$$\tan\theta = \theta. \tag{5.7}$$

In Snyder's extension [4] of their method to the magnetohydrodynamic problem he obtained

$$\mathrm{a}_n = -2\theta_n{}^2 - 2M^2. \tag{5.8}$$

This would lead to $\mathrm{a}_1 = -40.38 - 2M^2$, $\mathrm{a}_2 = -119.36 - 2M^2$ which may be compared with (5.5) and with the entries in Table II. Since (5.6) and (5.8) were obtained from equations linearized over the entire domain the agreement with Table II must be regarded as highly satisfactory.

The results given by Roidt and Cess [6] for the case of large R are in close agreement with those shown in Table II, and with equation (5.5). In terms of our variables they find that $\mathrm{a}_1(0) = -37.50$, $\mathrm{a}_1(2) = -45.68$, $\mathrm{a}_1(4) = -69.56$, $\mathrm{a}_1(6) = -107.90$.

Shercliff's earlier work([3], based on a totally different set of simplifying assumptions, also led to (5.8). He also took up the case of perturbations to ψ which were even functions of y, and for these obtained the exponents $-2n^2\pi^2 - 2M^2$.

We refer here also to the solution of the Navier-Stokes equations by Bodoia and Osterle [1]. Analysis of their results leads, for $M = 0$, to the estimate $\mathrm{a}_1 \approx -36$.

5.3 Small Hartmann Number

Consider the case $M \ll 1$. We develop (3.10), in powers of M retaining only the first few terms

$$\begin{aligned} Mg' - \frac{\alpha R_m}{2}\frac{My}{2}(1-y^2)\left\{1 + \frac{M^2}{20}(3+y^2)\right\} g \\ - \alpha\frac{3}{2}(1-y^2)\left\{1 + \frac{M^2}{60}(11+5y^2)\right\} h + \frac{2}{R_m}[h'' + \alpha^2 h] = O(M^4) \end{aligned} \tag{5.9}$$

$$\begin{aligned} \frac{2}{R}[g'''' + 2\alpha^2 g'' + \alpha^4 g] - \alpha\frac{3}{2}(1-y^2)\left\{1 + \frac{M^2}{60}(11+5y^2)\right\}\cdot[g'' + \alpha^2 g] \\ - \alpha\left\{1 + \frac{M^2}{10}(1+5y^2)\right\} g - \alpha\frac{My}{2}(1-y^2)\left\{1 + \frac{M^2}{20}(3+y^2)\right\}[h'' + \alpha^2 h] \\ + \frac{4M}{RR_m}[h''' + \alpha^2 h'] - \frac{2\alpha}{R}\cdot 3My\left\{1 + \frac{M^2}{30}(3+5y^2)\right\} h = O(M^4). \end{aligned} \tag{5.10}$$

If we had neglected all except the lowest order terms we should have obtained

$$h'' + \alpha^2 h - \tfrac{3}{2}\alpha R_M(1 - y^2)h = 0 \tag{5.11}$$

$$g'''' + 2\alpha^2 g'' + \alpha^4 g - \tfrac{3}{4}\alpha R\{(1 - y^2)(g'' + \alpha^2 g) + 2g\} = 0. \tag{5.12}$$

These equations, as also their boundary conditions, are now uncoupled and the eigenvalues of each can be obtained separately.

TABLE III

DOMINANT HYDRODYNAMIC AND MAGNETIC ROOTS AS FUNCTIONS OF R FOR $M = 1$, $R_m = 10^3$ [a]

R	$\alpha_2^{(M)}$	$\alpha_1^{(H)}$	$\alpha_1^{(M)}$	$\alpha_2^{(H)}$
0	−.0427481		−.0038010	
1000	(*) −.040629 ± .0046165i		−.0038577	
2000		−.0198082	−.0038638	
5000		−.0077894	−.0038961	
7000		−.0054742	−.0039565	
8000		−.0046701	−.0040572	
8200		−.0044945	−.0041128	
8300		−.0043810	−.0041685	
8340		−.0043067	−.0042201	
8348		−.0042666	−.0042556	
8400		−.0042465 ± .00010777i		
8600		−.0041920 ± .00022652i		
9000		−.0040904 ± .00033088i		
10000		−.0038719 ± .00040534i	(**)	
11000		−.0036932 ± .00037129i		
12000		−.0035445 ± .00025842i		
12500		−.0034790 ± .00014620i		
12700		−.0034543 ± .00004431i		
12720		−.003448	−.003455	
12800		−.0033530	−.0035315	
13000		−.0032507	−.0035866	
14000		−.0029434	−.0036785	
16000		−.0025369	−.0037355	
20000		−.0020100	−.0037759	
25000		−.0016009	−.0038072	
30000		−.0013312	(***) −.0038509 ± .00013817i	
40000		−.0009963	−.0037720	−.00295936

[a] In the range of R where some $\alpha_i^{(H)} \sim$ some $\alpha_i^{(M)}$ the two "interact" to produce a pair of complex roots. (*) Interaction between $\alpha_1^{(H)}$, $\alpha_2^{(M)}$; (**) Interaction between $\alpha_1^{(H)}$, $\alpha_1^{(M)}$; (***) Interaction between $\alpha_1^{(M)}$, $\alpha_2^{(H)}$. Note the insensibility of $\alpha_1^{(M)}$ to changes in R.

Let $\{\alpha_n^{[M]}\}$ be the eigenvalues of (5.11) with negative real part arranged in descending order of real part. We denote by $\{\alpha_n^{[H]}\}$ the corresponding set of eigenvalues of (5.12). The significance of these roots is that, for the uncoupled equations (5.11), (5.12), $\{\alpha_n^{[M]}\}$ represents the "magnetic roots" and $\{\alpha_n^{[H]}\}$ the "hydrodynamic roots".

When M is finite the equations (5.11), (5.12) have to be replaced by the coupled equations (3.10), (3.11). The eigenvalues of this pair can differ considerably from those corresponding to $M \approx 0$ but are still made up of two main sets. One set, to which we continue to refer as the "magnetic roots", denoted by $\{\alpha_n^{(M)}\}$, depend only slightly on R. The remaining roots, i.e. the "hydrodynamic roots" which we denote $\{\alpha_n^{(H)}\}$, are insensitive to changes in R_m. The only roots of both sequences which are substantially affected by the coupling are those magnetic roots which are close in value to hydrodynamic roots, and conversely. The effect of the coupling is to replace real roots by conjugate pairs of complex roots.

TABLE IV

LEADING MAGNETIC ROOTS FOR LIMITING CASE $M \to 0$

R_m	$\alpha_1^{[M]}$	$\alpha_2^{[M]}$
0	$-\frac{\pi}{2} = -1.5707963$	$-\frac{3\pi}{2} = -4.7123890$
0.0001	-1.5707637	-4.7123631
0.1	-1.538530	-4.686618
1	-1.277940	-4.461262
10	-0.357565	-2.79140
100	-0.0376819	-0.425182
1000	-0.00377035	-0.0428595

TABLE V

M	$A_1 = \lim_{R_m \to \infty} \alpha_1 R_m$	$A_2 = \lim_{R_m \to \infty} \alpha_2 R_m$
0	-3.7703	-42.8631
0.05	-3.77056	-42.86449
0.1	-3.77122	-42.86875
1	-3.85288	-43.51112
2	-4.05773	-46.1950
3	-4.299	-52.22
5	-4.69487	-82.376
10	-5.1766	

TABLE VI

Two Leading Roots for Various Values of R, R_m, M

M	R \ R_m	.0001	.1	1	10	100	1000
.1	.1	−1.5707637	−1.538538	−1.27800	−.357640	−.0376905	−.00377120
		−3.73436 ± 1.38690i	−3.73435 ± 1.38690i	−3.73426 ± 1.38690i	−2.792	−.4252	−.042865
	1	−1.5707637	−1.538538	−1.27800	−.357640	−.376905	−.00377120
		−3.60771 ± 1.39467i	−3.60770 ± 1.39467i	−3.60762 ± 1.39468i	−2.792	−.4252	−.042865
	10	−1.570764	−1.538541	−1.27802	−.357644	−.0376906	−.00377120
		−2.80096 ± 1.00842i	−2.80095 ± 1.00842i	−2.80087 ± 1.00843i	−2.793	−.42525	−.042865
	20	−1.570764	−1.538556	−1.27807	−.357649	−.0376906	−.00377120
		−1.91182	−1.91179	−1.91170	−1.91225	−.42526	−.042865
	100	−.375995	−.375995	−.376000	−.35844	−.0376911	−.00377120
		−1.15420	−1.15420	−1.15424	−.37512	−.376339	−.042866
	1000	−.0376479	−.0376479	−.0376479	−.0376484	−.037665 ± .0004127i	−.00377125
		−.115072	−.115072	−.115072	−.115073		−.037679
	10000	−.00376484	−.00376484	−.00376484	−.00376485	−.00376489	−.00387186 ± .000405i
		−.0115072	−.0115072	−.0115072	−.0115072	−.0115073	
1	.1	−1.5707644	−1.539238	−1.28369	−.364623	−.0385057	−.003852853
		−3.75326 ± 1.45860i	−3.75238 ± 1.45875i	−3.7444 ± 1.459i	−2.8726	−.43179	−.0435078
	1	−1.5707645	−1.539252	−1.28378	−.365656	−.0385061	−.003852857
		−3.62700 ± 1.4665i	−3.62616 ± 1.4667i	−3.61869 ± 1.4668i	−2.8768	−.43186	−.0435085
	10	−1.570765	−1.539469	−1.28514	−.365114	−.0385101	−.003852896
		−2.81414 ± 1.10713i	−2.81334 ± 1.10727i	−2.80661 ± 1.10784i	−2.9688	−.43269	−.0435158

1	20	−1.570766	−1.54045	−1.28924	−.36549	−.0385147	−.00385294
		−2.106733	−2.10519	−2.09902	−2.27873 ± .18787i	−.433867	−.0435240
	100	−.396560	−.396615	−.397163	−.40985 ± .06245i	−.0385543	−.00385330
		−1.1828	−1.1830	−1.1890		−.404119 ± .047092i	−.0435976
	1000	−.0396844	−.0396849	−.0396895	−.0397406	−.038707 ± .0040482i	−.00385772
		−.117673	−.117674	−.117681	−.117773		−.0406229 ± .004616i
	10000	−.00396847	−.00396848	−.00396852	−.00396899	−.00397409	−.0038719 ± .0004053i
		−.0117671	−.0117671	−.0117672	−.0117679	−.0117771	
5	.1	−1.570770	−1.54494	−1.33119	−.433978	−.0469075	−.0046948
		−4.13657 ± 2.42738i	−4.129455 ± 2.42854i	−4.06675 ± 2.43336i	−3.47784 ± 2.14028i	−.8723	−.082402
	1	−1.570770	−1.54497	−1.33144	−.434126	−.0469095	−.0046949
		−4.01405 ± 2.43314i	−4.00709 ± 2.43447i	−3.94574 ± 2.44094i	−3.58968 ± 2.10296i	−.91065 ± .04595i	−.08255
	10	−1.570771	−1.54535	−1.33431	−.435686	−.0469298	−.0046951
		−3.13235 ± 2.21162i	−3.12602 ± 2.21358i	−3.07021 ± 2.22665i	−2.62610 ± 2.08754i	−.87788 ± .21044i	−.08422
	20	−1.570771	−1.545932	−1.338533	−.437603	−.0469525	−.00469528
		−2.66254 ± 1.81063i	−2.65605 ± 1.81254i	−2.59965 ± 1.82573i	−2.14792 ± 1.76106i	−.82187 ± .30909i	−.086738
	100	−.898365	−.901365	−.94118	−.47000	−.047144	−.0046971
		−1.570770	−1.54588	−1.29577	−.90547	−.42928 ± .35502i	−.09054 ± .01914i
	1000	−.0869769	−.0869966	−.0871771	−.089347	−.053091	−.0047186
		−1.171038	−.177059	−.171253	−.174358	−.07990	−.043212 ± .035822i
	10000	−.00869520	−.00869539	−.00869717	−.0087151	−.0089312	−.005318
		−.0170969	−.0170971	−.0170990	−.017182	−.01742	−.007979

In Table III we show the dominant eigenvalues of each set for $R_m = 10^3$, $M = 1$ and a range of values of R. The "interaction" leading to the complex roots is seen clearly in this table.

We note that when $M = 0$ the magnetic roots are irrelevant and $\alpha_n = \alpha_n^{(H)} = \alpha_n^{[H]}$. Some of these roots were shown in Table I. (Note that (5.12) is identical with (5.1)). For small positive M the leading hydrodynamic root will become $\alpha_1^{(H)} = \alpha_1^{[H]} + O(M^2)$. However we now have to take into account also the magnetic roots. Hence α_1 will be $\alpha_1^{(H)}$ or $\alpha_1^{(M)}$ whichever has the larger real part. In case $\operatorname{Re} \alpha_1^{[H]} < \operatorname{Re} \alpha_1^{[M]}$, α_1 will change discontinuously as $M \to 0$. This situation is found to occur if R_m/R is not too small or, for any R_m, if $R < 24$.

We show in Table IV some of the magnetic roots in the limiting case $M = 0$ for various values of R_m.

5.4 Small R_M

We shall now consider the case $R_M \ll 1$. We see clearly from (3.10), (3.11) that $h(y)$ must be approximately of the form $A \cos \alpha y$ and so, in view of (3.12)

$$\alpha_n^{(M)} = -(n - \tfrac{1}{2})\, \pi + O(R_m) \tag{5.13}$$

Whether or not $\alpha_1^{(M)}$ will be the dominant eigenvalue will depend on the hydrodynamic roots, as already indicated. For $R \gg 1$, we can use (5.5) and it appears therefore that

(i) if $(37.62779 + 2M^2)/R < \pi/2$, the dominant root will still be given by (5.5), but

(ii) if $(37.62779 + 2M^2)/R > \pi/2$, the dominant root will now be $-\pi/2$.

5.5 Large Magnetic Reynolds Number

To deal with the case $R_M \gg 1$ we put $\alpha = \mathrm{A}/R_M$, $h = R_M\mathrm{H}$, $g = \mathrm{G}$. After a little rearrangement and keeping only terms of highest order in R_M the equations (3.10), (3.11) become

$$2\mathrm{H}'' + \mathrm{A}K(\cosh My - \cosh M)\mathrm{H} + M\mathrm{G}' + \tfrac{1}{2}\mathrm{A}K(\sinh My - y \sinh M)\mathrm{G} = 0 \tag{5.14}$$

$$2M\mathrm{H}''' + \mathrm{A}K(\sinh My - y \sinh M)\mathrm{H}'' - \mathrm{A}KM^2 \sinh My \cdot \mathrm{H} + \mathrm{G}'''' = 0. \tag{5.15}$$

We note incidentally that R does not appear in (5.14), (5.15) and so, as in paragraph 5.2, the eigenvalues depend only on M. We show in Table V the leading eigenvalues as a function of M for $R_m \gg 1$.

When R, R_m are both large the situation is as shown in Table III above.

5.6 General Description

After the above discussion we can now present (Table VI) the two leading eigenvalues corresponding to various values of M, R, R_M. This table should make clear the main features already mentioned. For example, we note that α_1 hardly changes as we go down a column, i.e. with change of R, until we come to the point where $\text{Re}\ \alpha_1^{(H)} < \text{Re}\ \alpha_1^{(M)}$ and α_1 then changes over from $\alpha_1^{(M)}$ to $\alpha_1^{(H)}$. We see also from Table VI that $\alpha_1^{(M)}$ can be approximated with reasonable accuracy in the range $0 \leqslant R_m \leqslant 1$, $0 \leqslant M \leqslant 5$ by

$$\alpha_1^{(M)} = -\tfrac{1}{2}\pi + (0.326 - 0.007M - 0.03R_m)\, R_m \tag{5.16}$$

[cf. (5.13)].

6. Comparison with Complete Solution

Elsewhere [2] we have reported the numerical solution of the complete hydrodynamic equations (2.1), (2.2) for the inlet region of a straight channel. Indeed part of the motivation for the work reported in the present paper was to check the accuracy of this numerical solution. From a study of the numerical results we were able in some cases to determine fairly accurately the exponential (real or complex) rate of approach to the limiting flow. We show in Table VII the results of this analysis for a number of sets of values of R, M, R_m, along with the theoretical values as determined from the linearized equations by the methods described above.

Determination of the eigenvalues from the complete numerical solution is fraught with difficulty. One difficulty is due to the interference of higher order terms which necessitates the examination of $\psi(x, y)$ for very large x. However in this region $\psi(\infty, y) - \psi(x, y)$ is small so that effects of round-off and truncation errors can be significant. This difficulty is particularly serious for $R < R_c$ when the solution is an oscillatory function of x and only the order of magnitude of $\alpha_1^{(H)}$ could be obtained. In the cases where it was possible to make the determination the agreement is seen to be good.

TABLE VII

COMPARISON WITH DOMINANT ROOTS AS ESTIMATED FROM NUMERICAL SOLUTION OF COMPLETE EQUATIONS

			Hydromagnetic roots obtained by methods of current paper		Exponent as obtained from complete numerical solution	
R	M	R_m	$\alpha_1^{(H)}$	$\alpha_1^{(M)}$	Fluid velocity Field	Magnetic Field
0	0		$-3.749 \pm 1.384i$		Order of	
0.5	0		$-3.677 \pm 1.391i$		magnitude	
5	0		$-3.154 \pm 1.297i$		of $-3 \pm i$	
20	0		-1.910100		-1.91	
50	0		-0.749171		-0.74	
100	0		-0.375787		-0.37	
200	0		-0.188075		-0.188	
500	0		-0.0752515		-0.075	
20	1	.0001	-2.1067	-1.570766	-2.1	-1.4
20	2	.0001	$-2.577 \pm 0.665i$	-1.570768	-3	-1.4
20	1	1	-2.0990	-1.28924	-2.2	-1.1
20	1	10	$-2.279 \pm 0.1879i$	-0.36549	-0.35	-0.35
20	1	50		-0.0769101	-0.076	-0.075
200	1	1	-0.1985204		-0.20	-0.2

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